Continuous-time Markov Chains.

We will need to generalize the def-n of discrete Markov chains. This requires a few ingredients.

Def-n. Let $t \in T$ be a parameter (i.e. time) and $S(t)$ a random variable for every t. The family of random variables $\{S(t) \mid t \in T\}$ is called a stochastic process.

Def-n. A stochastic process $\{S(t) \mid t \in T\}$ is said to satisfy the Markov property (MP) if for any $n > 0$ and $t_{i_0} < t_{i_1} < \dots < t_{i_n}$:

$$P\left(S(t_{i_n}) = s_{i_n} \mid S(t_{i_0}) = s_{i_0}, \dots, S(t_{i_{n-1}}) = s_{i_{n-1}}\right) =$$

$$= P\left(S(t_{i_n}) = s_{i_n} \mid S(t_{i_{n-1}}) = s_{i_{n-1}}\right).$$

Rmk: The property means that 'the future, given the present, does not depend on the past'.

Examples:

1. Let (S, P) be a Markov chain with $S = \{s_1, \dots, s_N\}$. Then $P\left(S(t_{i_n}) = s_{i_n} \mid S(t_{i_0}) = s_{i_0}, \dots, S(t_{i_{n-1}}) = s_{i_{n-1}}\right) =$

$$= P\left(S(t_{i_n}) = s_{i_n} \mid S(t_{i_{n-1}}) = s_{i_{n-1}}\right) = p_{i_{n-1} i_n},$$ hence the MC (S, P) has MP (here $T = \mathbb{Z}_{\geq 0}$ is discrete).

2. Consider the stochastic process $\{S(t)\}$ with $t \in T = [0; +\infty)$ and states $S = \{0, 1, 2, 3, \ldots\}$ (countably many states), s.t. $\{S(t) \mid t \in T\}$ has MP. Then $S(t)$ is called a <u>continuous-time MC</u>.

<u>Examples:</u>

(1) $T = \{0, 1, 2, \ldots\} = \mathbb{Z}_{\geq 0}$, $S = \mathbb{Z}$, $S(t_0) = 0$.

$$S(t_i) = \begin{cases} S(t_{i-1}) + 1, & p = 1/2 \\ S(t_{i-1}) - 1, & p = 1/2 \end{cases} \quad \text{(random walk)}.$$

<u>Exercise.</u> Show that this process has MP.

(2) $T = \mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$, $S = \{0, 1, \ldots, 10\}$.

$S(t_i)$ is uniform on $\{0, 1, \ldots, 10\}$, i.e.

$$P(S(t_i) = k) = \frac{1}{10} \text{ for } 0 \leq k \leq 10.$$

Assume, in addition, $S(t_i) = S(t_{i-1})$ (*)

Notice that given $S(t_i) = k$ for some i, condition (*) implies $S(t_j) = k$ for any $j \in \mathbb{Z}$. It follows that $\{S(t) \mid t \in T\}$ has MP.

(3) Same as (2), but instead of condition (*), impose

$$S(t_i) = S(t_{i-2}) \quad (**)$$

Let us show that this stochastic process does not have MP. Choose $n=2$, $t_0=0$, $t_1=1$ and $t_2=2$ with $S(0)=0$ and $S(1)=1$. Then $P\{S(2)=0 \mid S(0)=0, S(1)=1\}=1$ (since $S(2)=S(2-2)=S(0)$). However, $P\{S(2)=0 \mid S(1)=1\}=1/10$.

(4) $T=[0;+\infty)$, $S=\{0,1,\dots,10\}$, $S(t)$ is uniform on $\{0,1,\dots,10\}$ and $S(t)=S(t-1)$ for $t\geq 1$.

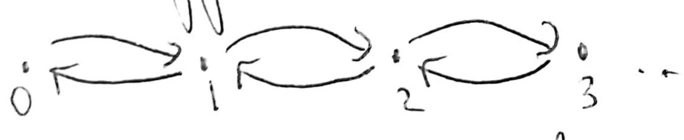
The process $\{S(t) \mid t\in T\}$ does not have MP. Choose $n=2$, $t_0=0.5$, $t_1=0.7$ and $t_2=1.5$ with $S(0.5)=1$, $S(0.7)=2$.

Then $P\{S(1.5)=1 \mid S(0.5)=1, S(0.7)=2\}=1$, but
(since $S(1.5)=S(1.5-1)=S(0.5)$)

$P\{S(1.5)=1 \mid S(0.7)=2\}=1/10$.

Birth-and-Death processes.

These are special cases of continuous-time MC where we impose an extra condition that transitions are allowed only between neighboring states. The transition from s_i to s_{i+1} is referred to as '<u>birth</u>' and in the opposite direction '<u>death</u>':



Let $\{S(t) \mid t\in[0;+\infty)\}$ be a stochastic process with MP.

Let $p_{ij}(t) = \mathbb{P}(S(t_0+t) = s_j \mid S(t_0) = s_i)$ be the family of transition probabilities from state s_i to s_j and $P(t) = \big(p_{ij}(t)\big)$ the corresponding family of transition matrices.

Rmk: we set $P(0) = I = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & \ddots \end{pmatrix}$ be the matrix with 1's on diagonal and 0's everywhere else. Notice, that as there are infinitely many states, $P(t)$ is infinite down ($\downarrow$) and to the right ($\rightarrow$), i.e.

$$P(t) = \begin{pmatrix} p_{00}(t) & p_{01}(t) & \cdots \\ p_{10}(t) & p_{11}(t) & \cdots \\ p_{20}(t) & p_{21}(t) & \cdots \\ \vdots & \vdots & \end{pmatrix}.$$

Def-n: the matrix $Q = \lim\limits_{\Delta t \to 0} \dfrac{P(\Delta t) - P(0)}{\Delta t} = \lim\limits_{\Delta t \to 0} \dfrac{P(\Delta t) - I}{\Delta t}$ is called the transition rate matrix.

In case $S(t)$ is a Birth-and-Death process

$$Q = \begin{pmatrix} -\lambda_0 & \lambda_0 & 0 & 0 & \cdots \\ \mu_0 & -\mu_0-\lambda_1 & \lambda_1 & 0 & \cdots \\ 0 & \mu_1 & -\mu_1-\lambda_2 & \lambda_2 & \ddots \\ \vdots & \vdots & \ddots & \ddots & \ddots \end{pmatrix},$$

where λ_i is the i^{th} birth rate and μ_i is the i^{th} death rate.

Def-n. The vector $\pi = (\pi_0, \pi_1, \pi_2, \dots)$ with all $\pi_n \geq 0$ and $\sum_{n=0}^{\infty} \pi_n = 1$ is called a stationary distr-n if $\pi_n = \lim_{t\to\infty} P_{in}(t)$ (for any i). Here we assume that the process has no absorbing states.

Thm 1. Let $S(t)$ be a Birth-and-Death process with birth rates $(\lambda_0, \lambda_1, \lambda_2, \dots)$ and death rates $(\mu_0, \mu_1, \mu_2, \dots)$. The stationary distr-n is given by

$$\pi_n = \frac{\lambda_0 \lambda_1 \dots \lambda_{n-1}}{\mu_0 \mu_1 \dots \mu_{n-1}} \pi_0, \text{ where } \pi_0 = \frac{1}{1 + \frac{\lambda_0}{\mu_0} + \frac{\lambda_1 \lambda_0}{\mu_1 \mu_0} + \dots}.$$

Proof: we start with the equality

$$P_{in}(t+\Delta t) = P_{in}(t) \cdot P(\Delta t),$$

subtracting $P_{in}(t)$ from both sides and dividing by Δt, get

$$\frac{P_{in}(t+\Delta t) - P_{in}(t)}{\Delta t} = P_{in}(t) \left(\frac{P(\Delta t) - \underbrace{P(0)}_{=I}}{\Delta t} \right).$$

Next, taking the limits when $t \to 0$, come up with

$$\lim_{t\to\infty} \frac{P_{in}(t+\Delta t) - P_{in}(t)}{\Delta t} = \boxed{P'_{in}(t) = P_{in}(t) \cdot Q} = \lim_{\Delta t\to 0} P_{in}(t) \frac{P(\Delta t) - P(0)}{\Delta t}$$

Rmk: the equations $P_{in}(t) = P_{in}(t) \cdot Q$ are called Kolmogorov forward equations.

Finally, taking limits when $t \to \infty$, obtain

$\lim_{t\to\infty} P'_{in}(t) = 0 = \pi \cdot Q$, where the first limit is zero, since $\lim_{t\to\infty} P_{in}(t) = \pi_n$ is a constant.

Finally, $\lim_{t\to\infty} \pi(t)\cdot Q = \pi \cdot Q$ must be equal to 0.

Recalling that $Q = \begin{pmatrix} -\lambda_0 & \lambda_0 & 0 & 0 & \cdots \\ \mu_0 & -\mu_0-\lambda_1 & \lambda_1 & 0 & \cdots \\ 0 & \mu_1 & -\mu_1-\lambda_2 & \lambda_2 & \ddots \\ \vdots & \vdots & \ddots & \ddots & \ddots \end{pmatrix}$, get

$$\pi Q = 0 \iff \begin{cases} \lambda_0 \pi_0 = \mu_0 \pi_1 \\ (\mu_0+\lambda_1)\pi_1 = \lambda_0 \pi_0 + \mu_1 \pi_2 \\ (\mu_1+\lambda_2)\pi_2 = \lambda_1 \pi_1 + \mu_2 \pi_3 \\ \cdots \end{cases}$$

<u>Rmk</u>: the system of equations above is called system of <u>balance eq-ns</u>.

Now, from the first balance eq-n $\pi_1 = \frac{\lambda_0}{\mu_0}\pi_0$, from the second $\pi_2 = \frac{\lambda_1\lambda_0}{\mu_1\mu_0}\pi_0$, etc.

<u>Exercise</u>. Show that $\pi_n = \frac{\lambda_{n-1}\cdots\lambda_1\lambda_0}{\mu_{n-1}\cdots\mu_1\mu_0}\pi_0$ (use induction)

Finally, as $\pi_0 + \pi_1 + \pi_2 + \ldots = \pi_0\left(1 + \frac{\lambda_0}{\mu_0} + \frac{\lambda_1\lambda_0}{\mu_1\mu_0} + \ldots\right) = 1$,

one has $\pi_0 = \dfrac{1}{\frac{\lambda_0}{\mu_0} + \frac{\lambda_1\lambda_0}{\mu_1\mu_0} + \ldots}$ (*).

Rmk: For the expression in (*) to converge, we need

$$1+\frac{\lambda_0}{\mu_0}+\frac{\lambda_1\lambda_0}{\mu_1\mu_0}+\dots<\infty.$$

For example it is sufficient to have $\frac{\lambda_i}{\mu_i}<q<1$, then

$$1+\frac{\lambda_0}{\mu_0}+\frac{\lambda_1\lambda_0}{\mu_1\mu_0}+\dots<1+q+q^2=\frac{1}{1-q}$$ is convergent geom

series.

Queues.

A basic queue model consist of three entities:

- arriving objects;
- a queue of arrived objects;
- (a) processing unit(s)

The Kendall notation

The standard notation for a queue model is

$A/S/c/b$, where

A is the pdf for arrival times

S is the pdf for service times

c is the number of parallel service channels

b is the system capacity restr-n (max. number of people in queue).

The M/M/1 model ($M/M/1/\infty$)

Customers are served one at a time, the length of the line is unbounded. The arrivals are according to a Poisson process and the service time distr-n is exponential (with parameters λ and μ).

Using Thm 1 we get that the fixed prob. vector is $w = (1-\rho, \rho(1-\rho), \rho^2(1-\rho), \dots)$, where $\rho = \frac{\lambda}{\mu}$, and one needs $\rho < 1$ for the expression in (*) to converge. $\boxed{\text{Rmk. The distr-n } \pi_n = \rho^n(1-\rho) \text{ is called geometric with param. } 1-\rho}$

Examples (1) (HW, page 23, #1). Suppose that 2 people arrive per hour and it takes an average of 10 minutes to serve a customer. Find the fixed probability vector for this queue.

Sol-n. This is an M/M/1 queue. Here $\lambda = 2$ and $\mu = 6$ (since $1 \text{ customer}/10 \text{ mins} = 6 \text{ customers}/\text{hour}$), so $\rho = \lambda/\mu = 1/3 < 1$.

Hence, $\pi_n = \rho^n(1-\rho) = \frac{1}{3^n} \cdot \frac{2}{3} = \frac{2}{3^{n+1}}$.

(2) HW, page 23, #2. Find the expected number of customers in the M/M/1 queue.

Sol-n. $L = \sum_{n=0}^{\infty} n\pi_n = \sum_{n=0}^{\infty} n \cdot \rho^n (1-\rho) = (1-\rho) \sum_{n=0}^{\infty} n\rho^n =$

$$= (1-\rho)\cdot\rho \frac{\partial}{\partial\rho} \sum_{n=0}^{\infty} \rho^n = (1-\rho)\cdot\rho \frac{\partial}{\partial\rho}\left(\frac{1}{1-\rho}\right) = \frac{\rho(1-\rho)}{(1-\rho)^2} = \frac{\rho}{1-\rho} =$$

$$= \frac{\lambda/\mu}{1-\lambda/\mu} = \frac{\lambda}{\mu-\lambda}$$

Example. (HW, p.24, #3).

M/M/3/3 queue, find π_i's.

Sol-n:

$$\begin{tikzcd} 0 \arrow[r, bend left, "\lambda"] & 1 \arrow[l, bend left, "\mu"] \arrow[r, bend left, "\lambda"] & 2 \arrow[l, bend left, "2\mu"] \arrow[r, bend left, "\lambda"] & 3 \arrow[l, bend left, "3\mu"] \end{tikzcd}$$

$$Q = \begin{pmatrix} -\lambda & \lambda & 0 & 0 \\ \mu & -\mu-\lambda & \lambda & 0 \\ 0 & 2\mu & -\lambda-2\mu & \lambda \\ 0 & 0 & 3\mu & -3\mu \end{pmatrix}$$

$$\pi Q = 0 \iff \begin{cases} \lambda\pi_0 = \mu\pi_1 \\ (\mu+\lambda)\pi_1 = \lambda\pi_0 + 2\mu\pi_2 \\ 3\mu\pi_3 = \lambda\pi_2 \end{cases}$$

$$\pi_1 = \underset{\substack{\| \\ \rho\pi_0}}{\frac{\lambda}{\mu}\pi_0}, \quad \pi_2 = \frac{\lambda^2}{2\mu^2}\pi_0 = \frac{\rho^2}{2}\pi_0 \quad \pi_3 = \frac{\lambda}{3\mu}\pi_2 = \frac{\rho^3}{6}\pi_0$$

$$\pi_0\left(1+\rho+\frac{\rho^2}{2}+\frac{\rho^3}{6}\right) = 1, \text{ so } \pi_0 = \frac{1}{1+\rho+\frac{\rho^2}{2!}+\frac{\rho^3}{3!}}.$$

Thm 2 (Little's Formula). Let L be the average number of customers in the system and W the average amount of time (including waiting and service) a customer spends in the system. Then

$$\boxed{L = \lambda \cdot W}$$

Let us give the idea of the proof. It is based on the <u>cost identity</u>. Imagine that each entering customer is forced to pay money (according to some rule) to the system. Then we would have the following identity:

$$(**) \qquad \text{average rate at which the system earns money} = \overset{\text{arrival rate}}{\lambda} \cdot \begin{pmatrix} \text{average amount of} \\ \text{money an entering} \\ \text{customer pays} \end{pmatrix}$$

<u>Proof (of $(**)$):</u> let $R(t)$ be the amount of money the system earned by time t, then

$$\text{l.h.s of } (**) = \lim_{t\to\infty} \frac{R(t)}{t} = \lim_{t\to\infty} \frac{N(t)}{t} \cdot \frac{R(t)}{N(t)} =$$

$= \lambda \lim_{t\to\infty} \frac{R(t)}{N(t)} =$ r.h.s of $(**)$, where $N(t)$ is the number of customers who entered the system by time t (inclusive)

To prove Little's formula, we use the following paying rule: each customer pays $1 per unit time he/she spends in the system. Then the average amount of money payed by a customer = average amount of time spent by a customer in the system (W). On the other hand, we can show that L = average rate at which the system earns money. Indeed, the amount of money earned by the system on the time interval $(t, t+\Delta t)$ is given by $X(t)\Delta t$, where $X(t)$ is the number of customers in the system at time t. Hence, the rate in question is $\lim\limits_{t\to\infty} \frac{\int_0^t X(s)ds}{t} = L$ (by def-n). Now the result follows from the cost identity.

<u>Rmk.</u> Let L_Q be the average number of customers waiting in queue and W_Q the average amount of time a customer spends in queue. Then $L_Q = \lambda \cdot W_Q$. The proof is completely analogous.

<u>Exercise.</u> Show the required modification to the paying rule.

Example. (HW, page 25, #1).

$M/M/1$ queue with $\lambda=2$ and $\mu=6$. Find L, W, L_Q, W_Q

Sol-n. We have shown that $L=\frac{\lambda}{\mu-\lambda}=\frac{2}{6-2}=\frac{1}{2}$ (page 7).

Using Little's f-la, we get $W=\frac{L}{\lambda}=\frac{1}{4}$.

Next, $W_Q=W-\frac{1}{\mu}=\frac{1}{4}-\frac{1}{6}=\frac{1}{12}$.

mean service time

Using Little's f-la again, $L_Q=\lambda W_Q=\frac{2}{12}=\frac{1}{6}$.

General f-las for the $M/M/1$ model:

$$\boxed{\begin{aligned} L&=\frac{\lambda}{\mu-\lambda}\\ W&=\frac{1}{\mu-\lambda}\\ W_Q&=\frac{\lambda}{\mu(\mu-\lambda)}\\ L_Q&=\frac{\lambda^2}{\mu(\mu-\lambda)}\end{aligned}}$$

Time Spent in the System.

Let T be the time a customer spends in the system. If there are n customers at the moment he/she arrives then T is the sum of the service times of $n+1$ customers.

Recall that the service time in $M/M/1$ queue has the exponential distr-n with pdf

$$f_X(x) = \begin{cases} \mu e^{-\mu x}, & x \geq 0. \\ 0 \end{cases}$$

This distribution also has the memorylessness property: $P(T > s+t \mid T > s) = P(T > t)$, $\forall s, t \geq 0$. $(*)$

We would like to find the pdf of r.v. T_n, the amount of time the customer spends in system given there are n customers in front.

It follows from $(*)$ that $f_{T_n}(t) = \frac{(\mu t)^n}{n!} \mu e^{-\mu t}$, $t > 0$.

<u>Exercise</u>. Show that for a collection of i.i.d. random variables $X_1, \ldots, X_n$ with pdf $f_{X_i}(t) = \mu e^{-\mu t}$ (exp. distr-ns with param. μ), one has $f_X(t) = \frac{(\mu t)^n}{n!} \mu e^{-\mu t}$, where $X = X_1 + \ldots + X_n$.

To obtain the distribution for T (time the customer spends in the system), we compute:

$$f_T(t) = \sum_{n=0}^{\infty} \frac{(\mu t)^n}{n!} \mu e^{-\mu t} \cdot \pi_n = \sum_{n=0}^{\infty} \frac{(\mu t)^n}{n!} \mu e^{-\mu t} \rho^n (1-\rho)$$

$$= \mu e^{-\mu t}(1-\rho) \sum_{n=0}^{\infty} \frac{(\lambda t)^n}{n!} = (\mu - \lambda) e^{(\lambda-\mu)t}, \ t > 0, \text{ which}$$

$\rho = \frac{\lambda}{\mu} \Rightarrow \rho \mu t = \lambda t$

is the exponential distr-n with parameter $\lambda - \mu$. It has the cdf $F_T(t) = P(T \le t) = \int_0^t (\mu-\lambda) e^{-(\mu-\lambda)s} ds =$

$= 1 - e^{-(\mu-\lambda)t}$.

<u>Rmk.</u> In particular, $W = \mathbb{E}(T) = \frac{1}{\mu - \lambda}$, as we have established before using Little's formula.

<u>Example. (page 26 #9)</u> For the $M|M|1$ system, where 3 people arrive each minute (on average) and it takes 15 sec. to serve a customer: ...

(a) Find L, L_Q, W, W_Q.

<u>Answer:</u> $\mu = 4$ customers/min, so $L = \frac{\lambda}{\mu - \lambda} = 3$,

$L_Q = \frac{\lambda^2}{\mu(\mu-\lambda)} = \frac{9}{4}$, $W = \frac{1}{\mu-\lambda} = 1$, $W_Q = \frac{\lambda}{\mu(\mu-\lambda)} = \frac{3}{4}$.

(b) give $f_T(t)$ and use it to find $P(T>1)$.

<u>Answer:</u> $f_T(t) = (\mu-\lambda)\, e^{-(\mu-\lambda)t} = e^{-t}$

$$P(T>1) = 1 - P(T\leq 1) = 1 - F_T(t) = 1 - \int_0^1 e^{-t}\,dt =$$

$$= 1 + e^{-t}\Big|_0^1 = 1 + (e^{-1} - 1) = e^{-1} = \frac{1}{e}.$$

(c) $$0.5 = \int_0^{median} e^{-t}\,dt = -e^{-t}\Big|_0^{median} = 1 - e^{-median}$$

$$e^{-median} = 0.5$$

$$-median = \ln(0.5).$$

$$median = -\ln(0.5).$$

#5. For M/M/2 queue. (page 24)

(a) find p_n

$$\begin{tikzcd} 0 \arrow[r, bend left, "\lambda"] & 1 \arrow[l, bend left, "\mu"] \arrow[r, bend left, "\lambda"] & 2 \arrow[l, bend left, "2\mu"] \arrow[r, bend left, "\lambda"] & 3 \arrow[l, bend left, "2\mu"] & \cdots \end{tikzcd}$$

$$\begin{cases} \lambda p_0 = \mu p_1 \\ (\lambda + \mu) p_1 = \lambda p_0 + 2\mu p_2 \\ (\lambda + 2\mu) p_2 = \lambda p_1 + 2\mu p_3 \\ \vdots \end{cases}$$

$$p_1 = \rho p_0$$

$$p_2 = \frac{\rho^2}{2} p_0 \qquad\qquad (\rho + \cancel{2}) \frac{\rho^2}{2} p_0 = \cancel{\rho^2 p_0} + 2 p_3$$

$$p_3 = \frac{\rho^3}{4} p_0$$

$$\vdots$$

Claim: $p_n = \dfrac{\rho^n}{2^{n-1}} p_0$

Proof: $(\lambda + 2\mu) p_{n-1} = \lambda p_{n-2} + 2\mu p_n \quad \Big| : \mu$ (ind-n assumption)

$$(\rho + \cancel{2}) \frac{\rho^{n-1}}{2^{n-2}} p_0 = \cancel{\frac{\rho^{n-1}}{2^{n-3}} p_0} + 2 p_n$$

$$p_n = \frac{\rho^n}{2^{n-1}} p_0 \qquad p_0 \underbrace{\left(1 + \rho + \frac{\rho^2}{2} + \frac{\rho^3}{4} + \cdots\right)}_{\displaystyle \| \atop \displaystyle 1 + \frac{\rho}{1 - \frac{\rho}{2}}} = 1$$

$$\Rightarrow p_0 = \frac{1}{\frac{1+\rho/2}{1-\rho/2}} = \frac{1-\rho/2}{1+\rho/2}$$

(b) $\lambda = 5/\text{hour}$, $\mu = 6/\text{hour}$ $\Rightarrow \rho = \frac{5}{6}$.

$$p_n = \frac{\rho^n}{2^{n-1}} \cdot \frac{1-\rho/2}{1+\rho/2} = \frac{5^n}{2^{n-1} \cdot 6^n} \cdot \frac{7/12}{17/12} = \frac{7}{17 \cdot 2^{n-1}} \cdot \left(\frac{5}{6}\right)^n .$$

(c) $$L = \sum_{n=0}^{\infty} n p_n = \sum_{n=0}^{\infty} n \frac{\rho^n}{2^{n-1}} \cdot \frac{1-\rho/2}{1+\rho/2} = \frac{2-\rho}{2+\rho} \cdot \rho \sum_{n=0}^{\infty} n \cdot \left(\frac{\rho}{2}\right)^{n-1} =$$

$$= \frac{\rho(2-\rho)}{2+\rho} \cdot \left(\frac{1}{1-\rho/2}\right)' = \frac{\rho(2-\rho)}{2+\rho} \cdot \frac{1}{2(1-\rho/2)^2} = \frac{\rho(2-\rho)}{(2+\rho)(2-\rho)(1-\rho/2)} =$$

$$= \frac{\rho}{(2+\rho)(1-\rho/2)} = \frac{\rho}{2(1+\rho/2)(1-\rho/2)} = \frac{\rho}{2(1-\rho^2/4)}$$

Example. (HW, p.24, #3).

$M/M/3/3$ queue, find π_i's.

Sol-n:

$$\begin{tikzcd} 0 \arrow[r, bend left, "\lambda"] & 1 \arrow[l, bend left, "\mu"] \arrow[r, bend left, "\lambda"] & 2 \arrow[l, bend left, "2\mu"] \arrow[r, bend left, "\lambda"] & 3 \arrow[l, bend left, "3\mu"] \end{tikzcd}$$

$$Q = \begin{pmatrix} -\lambda & \lambda & 0 & 0 \\ \mu & -\mu-\lambda & \lambda & 0 \\ 0 & 2\mu & -\lambda-2\mu & \lambda \\ 0 & 0 & 3\mu & -3\mu \end{pmatrix}$$

$$\pi Q = 0 \iff \begin{cases} \lambda\pi_0 = \mu\pi_1 \\ (\mu+\lambda)\pi_1 = \lambda\pi_0 + 2\mu\pi_2 \\ 3\mu\pi_3 = \lambda\pi_2 \end{cases}$$

$$\pi_1 = \underbrace{\frac{\lambda}{\mu}\pi_0}_{=\rho\pi_0}, \quad \pi_2 = \frac{\lambda^2}{2\mu^2}\pi_0 = \frac{\rho^2}{2}\pi_0 \qquad \pi_3 = \frac{\lambda}{3\mu}\pi_2 = \frac{\rho^3}{6}\pi_0$$

$$\pi_0\left(1+\rho+\frac{\rho^2}{2}+\frac{\rho^3}{6}\right) = 1, \text{ so } \pi_0 = \frac{1}{1+\rho+\frac{\rho^2}{2!}+\frac{\rho^3}{3!}}.$$

Problem. M/M/2/3 system, $\lambda = 3$, $\mu = 1$.

(a) Find p_0, p_1, p_2, p_3.

$$
\begin{tikzcd}
0 \arrow[r, bend left, "3"] & 1 \arrow[l, bend left, "1"] \arrow[r, bend left, "3"] & 2 \arrow[l, bend left, "2"] \arrow[r, bend left, "3"] & 3 \arrow[l, bend left, "2"]
\end{tikzcd}
$$

Balance eq-ns: $\begin{cases} 3p_0 = p_1 \\ (3+1)p_1 = 3p_0 + 2p_2 \\ (2+2)p_2 = 3p_1 + 2p_3 \end{cases}$

$p_1 = 3p_0$, $p_2 = 9/2\, p_0$, $p_3 = \frac{27}{4} p_2$

$p_0(1 + 3 + 9/2 + 27/4) = 1$.

$\frac{61}{4} p_0 = 1 \Rightarrow p_0 = \frac{4}{61}$, $p_1 = \frac{12}{61}$, $p_2 = \frac{18}{61}$, $p_3 = \frac{27}{61}$.

(b) Find L, L_Q, W, W_Q.

$L = 0 \cdot p_0 + 1 \cdot p_1 + 2 \cdot p_2 + 3p_3 = \frac{12 + 2 \cdot 18 + 3 \cdot 27}{61} = \frac{129}{61}$

Little's f-la, $W = \frac{129}{61 \cdot 3} = \frac{129}{183} = \frac{43}{61}$.

$W_Q = W - \frac{1}{\mu'} = \frac{43}{61} - \frac{61}{102} = 0.107$

mean waiting time

$\mu' = 1 \cdot p_1 + 2 \cdot p_2 + 2 \cdot p_3 = \frac{12 + 36 + 54}{61} = \frac{102}{61}$

Little's f-la, $L_Q = \lambda W_Q = 3 \cdot 0.107 = 0.321$.

Problem 6, page 26. Show that the exponential distr-n $f_X(x) = \lambda e^{-\lambda x}$ for $x \geq 0$, has the memoryless property:

$$P(X>a \mid X>b) = P(X>a-b).$$

Proof: $P(X>a \mid X>b) = \dfrac{P((X>a)\cap(X>b))}{P(X>b)} \underset{a>b}{=} \dfrac{P(X>a)}{P(X>b)}$

$$P(X>\alpha) = 1 - P(X\leq \alpha) = 1 - \int_0^{\alpha} \lambda e^{-\lambda x}\,dx = 1 + e^{-\lambda x}\Big|_0^{\alpha} =$$

$$= e^{-\lambda\alpha}.$$

Hence, $\dfrac{P(X>a)}{P(X>b)} = \dfrac{e^{-\lambda a}}{e^{-\lambda b}} = e^{-\lambda(a-b)}$.

$P(X>a-b) = e^{-\lambda(a-b)}$, ✓.

Problem 7, page 24. Suppose a food truck has one server and serves on average one person per 2 minutes. The arrival rate is $\frac{2}{n+1}$ people per minute, where n is the number of people in line.

(a) Find p_i's.

(b) Find L.

Sol-n: $\mu = 1$, $\lambda_n = \frac{4}{n+1}$ people / 2 mins

$$\begin{tikzcd} \bullet \arrow[r, bend left, "4"] & \bullet \arrow[l, bend left, "1"] \arrow[r, bend left, "2"] & \bullet \arrow[l, bend left, "1"] \arrow[r, bend left, "4/3"] & \bullet \arrow[l, bend left, "1"] \arrow[r, bend left, "1"] & \bullet \arrow[l, bend left, "1"] & \cdots \end{tikzcd}$$

(a) Notice that the balance eq-n at vertex n is

$$(\lambda_n + \mu_n)\, p_n = \lambda_{n-1} p_{n-1} + \mu_{n+1} p_{n+1} \quad \text{or}$$

$$\left(\frac{4}{n+1} + 1\right) p_n = \frac{4}{n} p_{n-1} + p_{n+1}, \text{ i.e.}$$

$$p_{n+1} = \left(\frac{4}{n+1} + 1\right) p_n - \frac{4}{n} p_{n-1} \quad (*)$$

Let $P(s) = \sum_{n=0}^{\infty} p_n s^n$ be the generating function for $(p_0, p_1, p_2, \ldots)$.

We want to see which functional eq-n on $P(s)$ corresponds to $(*)$.

Notice that $\int P(s)ds = \sum_{n=0}^{\infty} p_n \int s^n = \sum_{n=0}^{\infty} \frac{1}{n+1} p_n \cdot s^{n+1}$.

It follows that the functional eq-n we are looking for is $P(s) - p_0 + p_0 \cdot s = 4\int P(s)ds + sP(s) - 4s\int P(s)ds$ (☆)

<u>Check</u>: the coefficients of s^{n+1}:

$s^0: 0=0.$ ✓

$s^1: p_0 + p_1 = 4p_0 + p_0$, i.e. $p_1 = 4p_0$ ✓

$s^{n+1}, n \geq 1: p_{n+1} = 4 \cdot \frac{1}{n+1} p_n + p_n - 4 \cdot \frac{1}{n} p_{n-1} \leftrightarrow (*)$ ✓

Next we find the sol-n of functional eq-n (☆):

$$P(s) - sP(s) - p_0 + p_0 s = (4-4s)\int P(s)ds \quad | : (1-s)$$

$$P(s) - p_0 = 4\int P(s)ds$$

It is not hard to see that $P(s) = c \cdot e^{4s}$, where c is a constant. Since we know that $P(0) = p_0$, get

$P(0) = c = p_0$, hence $P(s) = p_0 \cdot e^{4s}$.

So $P(s) = \sum_{n=0}^{\infty} p_0 \cdot \frac{4^n}{n!} s^n$ and $p_n = \frac{4^n}{n!} p_0$.

Finally, as $p_0 \cdot \sum_{n=0}^{\infty} \frac{4^n}{n!} = 1$, we get $p_0 = e^{-4}$ and $p_n = \frac{4^n}{n!} e^{-4}$

(b) $L = \sum_{n=0}^{\infty} \frac{4^n}{n!} e^{-4} \cdot n = e^{-4} \sum_{n=0}^{\infty} \frac{4^n}{n!} \cdot n \overset{(!)}{=} e^{-4} \cdot 4 e^4 = 4.$

(!) Use that $x\frac{\partial}{\partial x}(e^x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} \cdot n$ and $x \frac{\partial}{\partial x}(e^x) = xe^x$.